Electromagnetic Descriptions of Waveguides and Transmission Lines

- Cutoff Frequency and Wave Impedance Explained -

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1. Introduction

In microwave/RF engineering, transmission line usually refers to any electrical conducting structure for the purpose of guiding electromagnetic (EM) waves. In this sense, transmission lines commonly used in practice include waveguide, coaxial cable and PCB based structures such as microstrip, stripline etc. Fig.1 illustrates some of them. In this article, however, we make a distinction between waveguide and transmission line as defined in the following sense: A waveguide is a hollow tube whose cross-section is completely enclosed within a metallic wall (Fig.1(a)) whereas a transmission line consists of at least two parallel conductors that support EM wave propagation (Fig.1(b) and (c)). This seemingly trivial difference in physical structure has profound effects on the characteristics of these structures in practical applications. Specifically, we fucus on two issues:

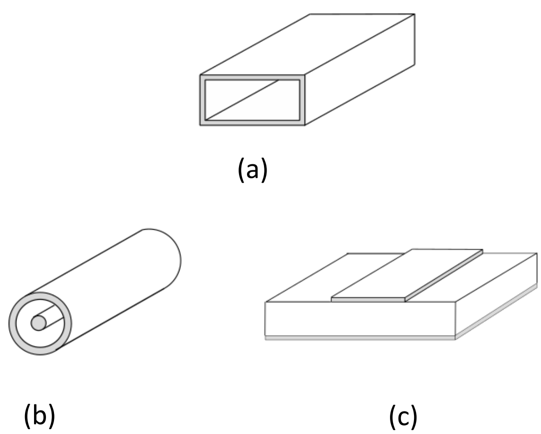


Figure. 1 (a) Waveguide. (b) Coaxial transmission line. (c) Microstrip transmission line.

1. **Frequency range**. A transmission line is generally considered broadband with the starting frequency at DC (the upper frequency limit of a transmission line depends on its specific physical properties) whereas a waveguide is of narrow-band nature and is characterized with a cutoff frequency below which EM waves cannot propagate.
2. **Impedance**. Characteristic impedance is a critical spec for transmission lines in practice but, in general, it is not a spec for waveguides. While a different type of impedance, namely wave

impedance, is often discussed in the context of waveguides, its use is limited to theoretical analyses rather than as a practical product specification. Then the question is how these two impedances are related.

In electrical engineering, circuit terms such as voltage, current and impedance are usually preferred in discussions, still electromagnetic description is necessary to understand, at a fundamental level, how these abovementioned differences come about. In this article we will use the principle of electromagnetism to explain the similarity and difference between the waveguide and transmission line in the context of practical applications. The subject matter in this article can be found in many standard textbooks on electromagnetism. However, due to the typical structure of such textbooks, these materials are often scattered in different chapters and under various subject headings. The intention here is to reorganize the materials so that the reader can easily see how a set of physics-based equations leads to conclusions regarding the two engineering issues outlined above. The focus of discussion is to shed some physical insight on the operation principles rather than any practical design guidance. To reduce the mathematical burden, we show derivations in more detail than most textbooks allow. Still, we assume readers have a working knowledge of wave equations, phasor notation and basic operation with complex numbers and differential equations. These subjects are generally covered in college courses in physics and engineering [1-3]. We also note that the SI units are used throughout this document. Some formulas in this article may appear different from those in other publications particularly older ones where the CGS system was used.

2. Equations Governing EM Wave Propagation

In general, the properties of electromagnetic (EM) waves are governed by Maxwell's equations. From engineering perspective, these equations answer two major questions: i) How EM waves are generated (radiation problem); ii) How EM waves propagate. In this article we are only concerned with the second question. Furthermore, we will limit our discussion to a specific subset of EM waves that satisfy two sets of equations presented in Eqs.1-3. All statements about EM waves throughout this article are limited to this subset and should be interpreted within this context. Specifically, we only consider EM wave behaviors in media where no electric charges and currents exit and permittivity $\varepsilon(=\varepsilon_r\varepsilon_0)$ and permeability $\mu(=\mu_r\mu_0)$ are scalar constants. Under these conditions, Maxwell's equations reduce to a set of homogeneous Helmholtz's equations for electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\nabla^2\boldsymbol{E} = -k_\omega^2\boldsymbol{E}, \quad \nabla^2\boldsymbol{H} = -k_\omega^2\boldsymbol{H} \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors and spatial dependent only. Leading to (1), the time dependence of the fields has been assumed to be $e^{j\omega t}$ in phasor representation [2,3], which is also referred to as time-harmonic solution in the literature. Note that the Laplacian operator ∇^2 in Eq.1 is not the same as another common vector operation $\nabla(\nabla\cdot)$, even though both yield a vector. Eq.4 explicitly shows how the Laplacian operator works on a vector in an xyz coordinate system.

A note on notation: in the discussions of waves, k is a common symbol for the wave number, also referred to as propagation constant, which is related to the wavelength λ as $k = 2\pi/\lambda$. Here the subscript ω in k_ω in Eq.1 indicates that this parameter is related to frequency as

$$k_\omega^2 = \omega^2\mu\varepsilon \tag{2}$$

It will be demonstrated in a number of examples in later sections that there are instances where k_ω is the same as the wave number but that is not a general rule. In the literature β is often used in the place of k_ω for the same purpose. In addition to Eq.1, $\boldsymbol{E}$ and $\boldsymbol{H}$ are related by a set of equations:

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H} \qquad \nabla\times\boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} \tag{3}$$

It's worth noting that the signs of the right-hand side of Eq.3 are based on our choice for the sign of the time factor ($e^{j\omega t}$) in the phasor representation. If $e^{-j\omega t}$ is chosen, the opposite sign convention should be used in Eq.3. Eqs.1 and 3 are the basic equations we will use in this article. The derivation of them from Maxwell's equations can be found in any textbook on electromagnetism [3-5] and is not repeated here.

Eqs.1 and 3 are in a 3-dimensional vector form. To obtain a useful solution, whether analytical or numerical, for a practical problem, we need to expand these equations in a coordinate system. There are three choices commonly employed, namely, Rectangular (also known as Cartesian), Cylindrical and Spherical coordinates. When the physical structure in question has a certain type of symmetry the choice is often obvious. For example, cylindrical coordinate is a natural choice for coaxial cables. Mathematically, analyses in the rectangular system are simplest. Since the purpose here is to explain the physics origin for the different behaviors of waveguides and transmission lines, we will minimize the mathematical complexity by limiting our discussion to the rectangular system. In such a coordinate, Eq.1 can be expanded into scalar equations as

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-k_\omega^2 V \tag{4}$$

where V represents any of the six field components: E_x, E_y, E_z, H_x, H_y and H_z. Under the assumption of separation of variables, a general solution for V has the form (see Eqs.9 and 10 in Appendix)

$$V(x,y,z)=\left(Ae^{jk_x x}+Be^{-jk_x x}\right)\left(Ce^{jk_y y}+De^{-jk_y y}\right)\left(Ee^{jk_z z}+Fe^{-jk_z z}\right) \tag{5}$$

or in trigonometric functions as

$$V(x,y,z)=(C_1\sin k_x x+C_2\cos k_x x)\left(C_3\sin k_y y+C_4\cos k_y y\right)(C_5\sin k_z z+C_6\cos k_z z) \tag{6}$$

where k_x, k_y, k_z are the wave numbers or the component of wave vector $\boldsymbol{k}$ in the respective directions and A through F and $C_i's$ are constants. We will use one of the forms based on the convenience of our analysis. Eq.5 or 6 indicates that the spatial variations of the field components are determined by $\boldsymbol{k}$. Hence we can see intuitively that all field components must have the same set of wave numbers. In addition, according to Eq. 11 in Appendix, k_x, k_y and k_z must satisfy

$$k_x^2+k_y^2+k_z^2=k_\omega^2=\omega^2\mu\varepsilon \tag{7}$$

As to the constants in Eqs.5 and 6, the boundary conditions for a given configuration (to be discussed in Sec. 3.1) will require some of them to be zero. A few examples in Section 4 illustrate this process.

As discussed in Appendix, each multiplier in Eq.5 represents a sum of two traveling waves in opposite directions (in phasor format). When both traveling waves exist the wave pattern is the so-called standing wave. In the following discussions, we are primarily concerned with wave propagation, that is, only one traveling wave presents. It is customary to assign $+z$ as the direction of wave propagation, which means $e^{-jk_z z}$ is the term that should be used in Eq.5 since our choice for time factor is $e^{j\omega t}$. Thus V can be expressed as

$$V(x,y,z) = V_t(x,y)\, e^{-jk_z z} \tag{8}$$

where $V_t(x,y)$ is the transverse component of V. Mathematically k_z can be a complex number with the imaginary part accounting for the wave decay in the z direction. Since our interest is wave propagation, k_z is always real throughout this article. In addition, the orientation of a coordinate system generally can vary in space (so-called local coordinates). We however assume in the following discussions that the coordinates are fixed. Under this assumption Eq.8 implies that the wave maintains a constant propagation direction (along z). In this convention, the phase of a wave is given by $\omega t - k_z z$ and the wave propagation (the movement of wave front) is fully characterized by the wave number k_z [2]. More discussions on this topic are provided in Sec. 3.4.

Since the z dependence of a field component is given in Eq.8, the derivative $\partial/\partial z$ can be replaced with $-jk_z$. Then we can write Eq.3 explicitly in six field components as

$$\begin{aligned}
&\frac{\partial E_z}{\partial y} + jk_z E_y = -j\omega\mu H_x \quad (a) &&\frac{\partial H_z}{\partial y} + jk_z H_y = j\omega\varepsilon E_x \quad (b)\\
&-jk_z E_x - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y \quad (c) &&-jk_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (d)\\
&\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (e) &&\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad (f)
\end{aligned} \tag{9}$$

This set of equations determines the interdependence of the field components. Before applying them to waveguides and transmission lines, which are the configurations of primary interest of this article, it is instructive to first consider a case of plane wave that has the simplest wave pattern. The reason for this excise is to illustrate how the wave pattern evolves as more spatial restrictions due to boundary conditions are imposed. As the name indicated, the fields of a plane wave are in a plane that is perpendicular to the direction of propagation. In our chosen coordinate system, this implies that $E_z =$

$H_z = 0$. To obtain the expressions for the other four components, we first derive E_x from an operation of $\mathrm{Eq.}\,9(c) \times k_z - \mathrm{Eq.}\,9(b) \times \omega\mu$:

$$E_x = \frac{j}{k_z^2 - k_\omega^2}\left(k_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right) \tag{10a}$$

Using a similar technique we obtain the rest field components:

$$E_y = \frac{j}{k_z^2 - k_\omega^2}\left(k_z \frac{\partial E_z}{\partial y} - \omega\mu \frac{\partial H_z}{\partial x}\right) \tag{10b}$$

$$H_x = \frac{-j}{k_z^2 - k_\omega^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial y} - k_z \frac{\partial H_z}{\partial x}\right) \tag{10c}$$

$$H_y = \frac{j}{k_z^2 - k_\omega^2}\left(\omega\epsilon \frac{\partial E_z}{\partial x} + k_z \frac{\partial H_z}{\partial y}\right) \tag{10d}$$

Now consider Eq.10a. Since $E_z = H_z = 0$, to have a nonzero E_x, we must have $k_z^2 - k_\omega^2 = 0$ ($E_x = 0/0$ simply means it is a yet to be determined constant), which in, turn according to Eq.7, leads to $k_x = 0$ and k_y. So for plane waves, Eq.9 (a)-(f) reduce to

$$\begin{aligned} &k_z E_y = -\omega\mu H_x \quad (a) \qquad k_z H_y = \omega\varepsilon E_x \quad (b) \\ &k_z E_x = \mu\omega H_y \quad (c) \qquad -k_z H_x = \omega\varepsilon E_y \quad (d) \\ &H_z = 0 \quad (e) \qquad E_z = 0 \quad (f) \end{aligned} \tag{11}$$

They can be further simplified if we choose the direction of $\boldsymbol{E}$ as x direction. In such a case $E_y = 0$. Then according to Eq.11(a) $H_x = 0$. Hence, Eq. 11(b) gives:

$$\frac{E_x}{H_y} = \frac{k_z}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \tag{12}$$

since $k_z^2 = k_\omega^2 = \omega^2\mu\varepsilon$. It can be easily verified that regardless of the choice of the coordinate system $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to each other and their ratio is given by Eq.12. In fact, η defined in Eq.12 is a special case of a more generic parameter called wave impedance. We will discuss this topic in more detail in the next section.

The plane wave just studied only occurs in an unbounded and uniform medium. What we are interested in this article is the characteristics of so-called guided waves where waves propagate in media bounded in a certain manner by electrically conducting walls. Therefore, the boundary conditions between conductor walls and dielectric media must be included in our analysis. In the following sections we will first outline the boundary conditions in such situations. Then several other concepts that are key to our discussions will be reviewed. In Sec.4 we will apply them to several physical structures. We conclude this section by a note: In a more general definition, a plane wave does not need to be in a fixed propagation direction [5]. Nevertheless, the term of plane wave in this article refers to the specific wave pattern outlined above.

3. Review of Some Background Information

3.1 Boundary Conditions

As in the case of a uniform space, the EM wave behaviors at various boundaries are completely determined by Maxwell's equations and auxiliary equations. In line with our intention to make discussions focused, we will skip the derivation and simply state the following boundary conditions. Detailed discussions can be found in [3-5].

At the surface of a PEC (perfect electric conductor where conductivity $\sigma \to \infty$), the tangential component of the electric field E_t and the normal component of the magnetic field H_N vanish. In equation form, these conditions are:

$$E_t = 0;\ H_N = 0 \tag{13}$$

For the purpose of this article, we only need the boundary condition on the E field.

3.2 Propagation Modes

For the reason that will be clear later, the propagation characteristic of an EM wave is classified by its "mode" which is defined in terms of which longitudinal components, E_z and H_z, being zero. Specifically, if both E_z and H_z are zero, the wave pattern is called TEM mode (meaning both electric and magnetic fields are only in transverse plane). The plane wave discussed in Sec. 2 is in TEM mode because of Eq.11(e)(f).

By the same token, we have TE mode and TM mode when $E_z = 0, H_z \neq 0$ and $H_z = 0, E_z \neq 0$ respectively. In Sec.4, we will show that for the TE and TM modes k_x and k_y are limited to a series of discrete values due to the boundary-condition requirements. Thus, each propagation mode is further classified by the allowed $k_x\ k_y$ values that are indexed by a pair of integers, m and n (see Eq.36 for the formula), for example TE_{01} and TM_{11}, etc. In practical applications a single mode is generally desirable for various reasons (a topic beyond the scope of this article). That is why the usable frequency range of a waveguide or a transmission line is usually specified according to the range in which only the lowest mode can propagate. For the same reason this mode is often referred to as the dominant mode.

3.3 Cutoff Frequency

We rewrite Eq.7 as

$$k_z^2 = k_\omega^2 - \left(k_x^2 + k_y^2\right) = \omega^2 \mu\varepsilon - \left(k_x^2 + k_y^2\right) \tag{14}$$

Eq.14 implies that unless both k_x and k_y are zero there exists a minimum frequency below which k_z becomes imaginary. This minimum frequency is called cutoff frequency as a traveling wave is no longer allowed below it. From Eq.14, the cutoff frequency ω_c is given by

$$\omega_c^2 = \frac{1}{\mu\varepsilon}\left(k_x^2 + k_y^2\right) \tag{15}$$

When both k_x and k_y are zero the cutoff frequency is zero too and k_ω is the same as wave number k_z. The plane wave discussed earlier is such a case. In contrast, waveguides, as will be shown later, must have at least one nonzero value of k_x and k_y. The smallest value of k_x or k_y sets the cutoff frequency for the waveguide to operate. In the literature, k_z is commonly expressed in terms of ω_c:

$$k_z = \sqrt{\mu\varepsilon(\omega^2 - \omega_c^2)} = \omega\sqrt{\mu\varepsilon\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]} = \frac{\omega}{v}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{16}$$

where $v = 1/\sqrt{\mu\varepsilon}$ is the wave velocity in the medium. Next subsection provides more discussions on the topic.

3.4 Phase and Group Velocities and Wave Dispersion

Eq.16 has another significant implication regarding the EM wave characteristics, that is, wave dispersion. To explore this topic, we rewrite the function for a z-directed traveling wave using the same notation as in Section 2:

$$\psi(z,t) = Ae^{j(\omega t - k_z z)} \tag{17}$$

where A is a constant and $\omega t - k_z z$ is the phase. From any introductory text on waves (e.g. [2]), we know ω and k_z in Eq.17 are related to frequency and wavelength by $f = \omega/2\pi$ and $\lambda = 2\pi/k_z$. Furthermore, the velocity of the wave is given by

$$v_p = f\lambda = \frac{\omega}{k_z} \tag{18}$$

The velocity in Eq.18 is called phase velocity (hence the index "p") since it represents the velocity at which a point of phase moves. To see this, let the phase be a constant c, that is, $\omega t - k_z z = c$. For a time change $\mathrm{d}t$, the corresponding spatial change to maintain the constant phase is $\mathrm{d}z = (\omega/k_z)\mathrm{d}t$. Thus, this constant-phase point moves at the speed given by Eq.18. We emphasize that it is the phase velocity here because soon we will introduce another velocity, namely, group velocity.

Eq.18 indicates that the three basic physical quantities, frequency, wavelength and phase velocity, of a wave must satisfy this simple relationship but it does not address whether they are interdependent. In most treatments of wave propagation problems, frequency is considered independent since it is determined by the wave source. Then the question is whether v_p (or λ) is independent of the frequency. For EM waves in free space the answer is yes: all EM waves regardless of frequency travel at a constant speed, the so-called the speed of light. However, when media or certain spatial constraints (the case that will be discussed in Section 4) are present, v_p is generally dependent of frequency. This is important in many applications particularly in communications where a microwave/RF signal is modulated. A modulated signal occupies a certain bandwidth, which mathematically can be modeled as a function that is a superposition of a number of frequency components. Then if the phase velocity varies with frequency a distortion in wave form may occur since different frequency components no longer move in a synchronized manner. This phenomenon is known as dispersion. For the case of a wave packet, which is often used to illustrate the effect of wave dispersion, it can be shown that the wave packet spreads over time. While the concept of wave spreading is perhaps not too difficult to visualize conceptually, the rigorous mathematical treatment is a little involved. For readers who want to dig deeper into this subject mathematically, [6] is an excellent reference for wave packet spreading (Note: quantum mechanics is not required for this part of analysis despite of the book title).

Next, we examine the conditions that lead to dispersion. We observe from Eq.18 that v_p being dependent on frequency is equivalent to k_z being a nonlinear function of ω. Then Eq.16 indicates that if the function $\sqrt{\mu\varepsilon[1 - (\omega_c/\omega)^2]}$ has any dependence on ω dispersion occurs. We can further breakdown the mechanisms for dispersion into two categories: The first is a medium characterized by

$\varepsilon(\omega)$, that is, ε is frequency dependent. In most applications particularly in optics this is the cause for dispersion. The second mechanism is a nonzero ω_c, which is in fact the primary interest of our discussion. In this article we will assume ε is a frequency-independent constant and focus on the conditions that lead a nonzero ω_c.

In discussions of dispersion effect, the group velocity is an important concept. It is defined as

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_z} \tag{19}$$

We first note that when dispersion occurs the group velocity is not the same as phase velocity. This can be seen from the following equation:

$$\frac{\mathrm{d}v_p}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega}{k_z}\right) = \frac{1}{k_z}\left(1 - v_p\frac{\mathrm{d}k_z}{\mathrm{d}\omega}\right) = \frac{1}{k_z}\left(1 - \frac{v_p}{v_g}\right) \tag{20}$$

Clearly when $\mathrm{d}v_p/\mathrm{d}\omega \neq 0,\ \ v_g \neq v_p$. The physical significance of v_g is this [3,4]: it represents the velocity at which energy and information are transmitted in many dispersive media (exceptions do occur). In contrast, the phase velocity is the speed of the phase at a single frequency. Detailed discussion on this topic is beyond the scope this article.

Another criterion for dispersion, especially in the context of wave packet spreading, is that a non-zero second order derivative of k_z, that is, $k_z^{"} = \mathrm{d}^2k_z/\mathrm{d}^2\omega \neq 0$, causes dispersion. The equivalence of $v_p' \neq 0$ and $k_z^{"} \neq 0$ can be easily verified by taking a second derivative of Eq.18 after rewriting it as $k_z = \omega/v_p$.

Now we come back to Eq.16 and use it to write explicitly v_p and v_g as a function of ω. As a reminder, the velocity in the medium, v, is given by:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{21}$$

Physically v is the velocity for a wave in an unbounded medium characterized with a constant ε without any dispersion effect. In the free space it is the speed of light. Then using Eq.16, we have the expression for v_p:

$$v_p = \frac{\omega}{k_z} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} = \frac{v}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} \tag{22}$$

Eq.22 shows that v_p has two independent factors: v is solely a property of the medium and the other is a function of ω_c/ω which is determined by the physical structure of the guided waves (see next section for more discussions). For v_g, we notice that Eq.16 is in the form of $k(\omega)$ rather than $\omega(k)$. So, it is easier to evaluate $1/v_g$ than v_g.

$$\frac{1}{v_g} = \frac{\mathrm{d}k_z}{\mathrm{d}\omega} = \sqrt{\mu\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\sqrt{\omega^2 - \omega_c^2}\right) = \sqrt{\mu\varepsilon}\frac{1}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \tag{23}$$

Thus,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_z} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = v\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \tag{24}$$

Obviously when $\omega_c = 0, v_p = v_g = v$. Furthermore, Eqs.18 and 24 show $v_p \geq v$ and $v_g \leq v$. They are the mathematical expression for a well-known fact that the phase velocity can be greater than the speed of light while the group velocity is always smaller than the speed of light. In addition, the effect of ω_c can also be expressed in wavelength by introducing the waveguide wavelength which is a parameter often seen in the literature:

$$\lambda_g = \frac{2\pi}{k_z} = \frac{\lambda}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \tag{25}$$

where $\lambda = v/f$ which is the wavelength in the same medium but unbounded. The index g here is for "guide" rather than "group". It is actually related to the phase velocity by $v_p = \lambda_g f$. Note $\lambda_g \geq \lambda$ is a direct result of $v_p \geq v$. We summarize the preceding discussion as follows:

- Frequency f, wavelength λ and phase velocity v_p are always related as

$$f\lambda = v_p$$

- In an unbounded uniform medium, v_p is a frequency independent constant, resulting in a simple relationship between f and λ as $f \propto 1/\lambda$.
- When dispersion occurs due to ω_c, v_p is given by Eq.22 and the wavelength by Eq.25.
- Group velocity is the speed for information transmission. It may or may not be the same as the phase velocity depending on the dispersion condition.

Finally, we observe that the dispersion effect under our assumed condition is entirely due to the factor $\sqrt{1-(\omega_c/\omega)^2}$. This function is plotted against ω/ω_c in Fig. 2. As can be seen from the plot, when ω moves away from ω_c, the function becomes less sensitive to the frequency. In practice the specified frequency range for a waveguide usually starts at about 25% above the corresponding cutoff frequency of the waveguide. More detailed discussion on this topic is in Sec. 4.3.

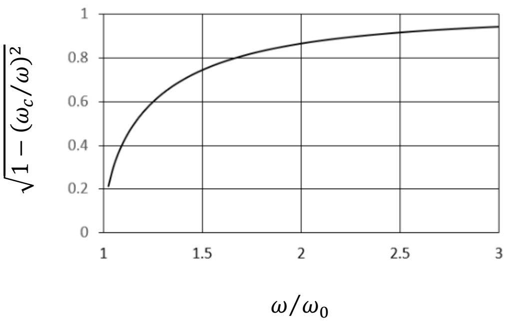


Figure. 2 $\sqrt{1-(\omega_c/\omega)^2}$ vs. ω/ω_c

3.5 Wave Impedance

As a general principle according to Maxwell equations, $\boldsymbol{E}$ and $\boldsymbol{H}$ always coexist in a time varying EM field. While their amplitudes depend on many factors, their ratio is usually uniquely determined by the medium. In the case of plane waves, we have shown this ratio is given by Eq.12. In fact, the wave impedance is defined in a more general term as follows [3-5]:

For a propagating EM wave, its wave impedance, denoted as η, is the ratio of the electric field to the magnetic field in the transverse plane

$$\eta = \frac{|\boldsymbol{E}_{\text{transverse}}|}{|\boldsymbol{H}_{\text{transverse}}|}$$

The reader may wonder why the wave impedance is defined in transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ instead of $\boldsymbol{E}$ and $\boldsymbol{H}$ themselves. This can be explained by the Poynting theorem [3-5] which states that the total power of an EM field flowing through a surface area S is given by:

$$\iint_S \boldsymbol{P}\cdot\boldsymbol{n}\,\mathrm{d}s$$

where $\boldsymbol{P}$ is the Poynting vector: $\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H}^*$ and $\boldsymbol{n}$ is the unit vector in the perpendicular direction of $\mathrm{d}s$. $\boldsymbol{n}$ is also the propagation direction in our case. Then according to the properties of vector products, only the parallel (to $\boldsymbol{n}$) component of $\boldsymbol{P}$, $P_{\parallel}$, contributes to the integral hence to the power flow in the wave propagation direction. In turn only the transverse (also with respect to $\boldsymbol{n}$) field components contribute to $P_{\parallel}$. This explanation also reveals why η is termed as impedance which is typically a circuit parameter: Recall that in electric circuits voltage V and current I are two basic parameters. The complex power delivered to a load is VI^* and the load impedance is calculated by V/I. Thus, the analogy between E, H and V, I is evident. Moreover, the wave impedance is an especially useful concept in electromagnetic field analyses, particularly in radiation problems, because in a uniform medium electric and magnetic fields are simply related by a constant. As a result, in many cases a solution needs to be worked out only for one field based on the convenience for a given condition rather than for both fields separately.

The wave impedance of a guided wave is generally dependent on the medium and physical structure. To clearly see their effects, we introduce three symbols for the wave impedance in three different conditions. We will use η_ε for the wave impedance of a plane wave which is given by

$$\eta_\varepsilon = \sqrt{\frac{\mu}{\varepsilon}} \tag{26}$$

here ε in the index indicates that the wave impedance in this case is solely dependent on the property of its medium. For the same reason, this wave impedance is also referred to as the intrinsic impedance. Following the convention in the literature a special symbol η_0 is used for the wave impedance in free space, that is,

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

When a specific propagation mode is under consideration, an index is added to indicate the mode, e.g. η_{TE}, etc.

The wave impedance is sometimes treated in the literature as the same as characteristic impedance of transmission lines. As will be clarified later, these two impedances are related but generally are not the same. Finally, Eq.26 is only applicable to dielectric media, that is, the conductivity is 0. For media with finite conductivity the wave impedance is a function of conductivity as well.

3.6 Characteristic Impedance of Transmission Lines

A transmission line is used in an electrical circuit for two main purposes: i) as a circuit element for certain functions such as impedance matching, filtering, etc. and ii) as an interconnection between components. For both cases, its characteristic impedance is a critical circuit parameter. In this subsection, we first review how the transmission-line impedance is derived, and in the next section we apply the EM theory outlined in Sec. 2 to analyze the wave behaviors in several physical structures, aiming to show how the wave impedance and the transmission line impedance are related for a specific type of transmission line, namely microstrip.

Fig. 3 shows an equivalent circuit for a lossless transmission line. The distributed inductance and capacitance are denoted as *L* and *C* in the figure. It turns out that equations for voltage $V(x,t)$ and current $I(x,t)$ for this circuit model are one-dimension Helmholtz equation. The derivation can be found in any textbook on RF engineering, for example [7-9]. Then the general solutions for them have the same functional form as that in Eq.5 (with only one variable), that is, it is a sum of two traveling waves in opposite directions.

$$V(x) = V_0^+ e^{-jkx} + V_0^- e^{jkx} \tag{27}$$

$$I(x) = \frac{V_0^+}{Z_0} e^{-jkx} - \frac{V_0^-}{Z_0} e^{jkx} \tag{28}$$

where $k^2 = LC$ and Z_0 is the characteristic impedance for the line and given by

$$Z_0 = \sqrt{\frac{L}{C}}$$

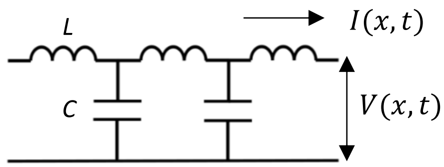


Figure 3 Equivalent circuit model for transmission lines.

So far Z_0 is just another parameter defined in two given circuit parameters. To see its physical significance, consider a case where the transmission line is terminated with a resistance R. Choosing $x = 0$ at the termination point, from Eqs. 27 and 28, we have

$$V(0) = V_0^+ + V_0^-$$

$$I(0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

On the other hand, the Ohm's law requires

$$\frac{V(0)}{I(0)} = R$$

Thus

$$Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} = R \tag{29}$$

It is immediately clear from Eq.29 that when $Z_0 = R, V_0^- = 0$. Since the reflected wave V_0^- is generally undesirable in RF circuits [10], this explains why the characteristic impedance is the most critical parameter in a transmission line design. It also explains why two transmission lines with drastically different physical structures can be connected together without causing much disruption to signal transmission as long as they have the same characteristic impedance.

Finally, it should be noted that for a traveling wave in the z direction, $V_0^- = 0$, then $Z_0 = V(x)/I(x)$, which will be the bridge between the EM-wave description (E and H) and the circuit-parameter (V and I) description for the transmission-line impedance (Sec. 4.2).

4. Electromagnetic Descriptions of EM Waves in Three Physical Structures

With the materials developed in the preceding sections we are ready to examine the behaviors of EM waves in three physical structures illustrated in Figs. 4-6. They are: parallel conducting planes, microstrip line and rectangular waveguide.

4.1 Parallel Conducting Planes

The geometry and the chosen coordinate system of this structure are those of Fig. 4. The wave is confined in the y-direction by two infinite parallel conducting planes. Note that this configuration is sometimes called "Parallel-plane waveguide" in the literature. In the context of this article, the name of waveguide is reserved for a tube-like structure where the EM wave's cross-section ($x - y$ plane) is completely enclosed by a conducting wall (see Fig. 6). In this narrow sense, this structure is not a waveguide.

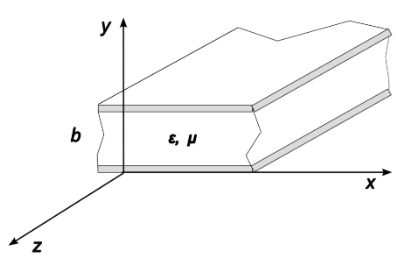


Figure 4 Parallel Conducting Planes

As discussed earlier we will seek a solution with the lowest cutoff frequency. As the case of plane waves, we try the TEM mode first, that is, $E_z = H_z = 0$. For the transverse components, unlike the plane wave where the $\boldsymbol{E}$ field can be in any direction on the $x - y$ plane, now E_x is forbidden, else the boundary condition, Eq.13, at $y = 0$ and b planes (where E_x is tangential to the planes) is not satisfied. The rest results developed for the plane wave are still valid. Thus, the wave solutions in this case are

$$E_y = E_0 \text{ (constant)}, \; H_x = -\frac{E_0}{\eta_\varepsilon}$$

$$E_z = 0,\ E_x = 0,\ H_z = 0,\ H_y = 0 \tag{30}$$

$$k_x = k_y = 0, \qquad \omega_c = 0$$

Thus, comparing with the plane wave, the TEM mode is still allowed for this configuration and the wave impedance is the same as that in Eq.26. The only effect of the two conducting planes is that the electric field now is limited to the y-direction.

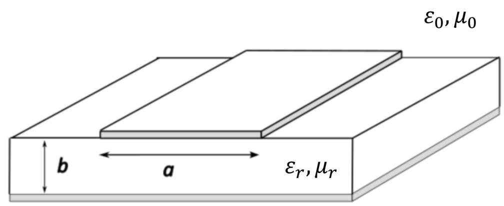
4.2 Microstrip transmission line

In the preceding case the width (x direction) is infinite for both planes. If we reduce the width of one of the planes to a finite value a, the structure becomes microstrip line which is one of the most commonly used PCB transmission lines in practice [9, 11]. In this configuration the $\boldsymbol{E}$ field at the edge is no longer confined in the dielectric region but rather leaks into air as illustrated in Fig. 5. Thus, strictly speaking in this structure the TEM mode is not valid anymore because the EM fields are not confined in a uniform medium. But if the thickness of the dielectric layer, b, is considerably smaller than both the wavelength λ and the strip width, a, in most region underneath the top metallic strip the field pattern is still approximately the same as the case of parallel conducting planes. As such the basic features discussed in Sec. 4.1 are expected to still hold. For this reason, the wave pattern in a microstrip transmission line is referred to as quasi-TEM mode [9]. Then it appears reasonable that for the purpose of this article (getting physical insight into the origin of the difference between the waveguide and transmission line) we will ignore the fringing effect at the edges. Under this approximation the field components in the space between $x = 0$ and $x = a$ are the same as these obtained in Eq.30. With this established, we now attempt to link the field components E and H to circuit variables V and I.

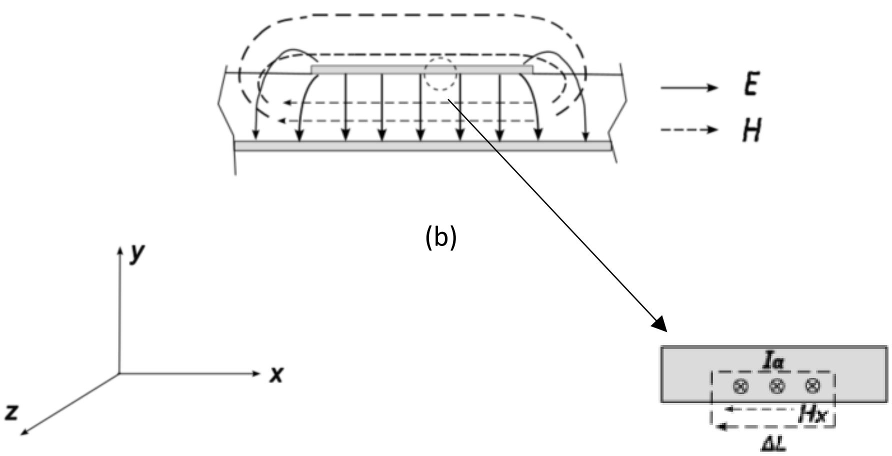
Referring to Fig. 5, it is straightforward to relate the $\boldsymbol{E}$ field to V between the two conductor planes by using the [2, Sec.23-2]

$$V = \int_0^b E_y \, dy = bE_0$$

The sign is not important here and thus ignored. The calculation for the magnetic field is less obvious. We work out the detail below.



(a)



(b)

Figure 5 (a) Microstrip transmission line. (b) Transverse field patterns.

Insert: magnetic field and surface current.

The ampere's law in integral form is [2, Sec.28-3]]

$$\oint_L \boldsymbol{H} \cdot \boldsymbol{dl} = I \tag{31}$$

where I is the total current through the surface enclosed by the path L. Applying Eq.31 to a cross section at the conductor-dielectric boundary as shown in the insert of Fig. 5 and assuming that the loop width is infinitesimally small and the magnetic field inside the conductor is zero (due to PEC assumption), we have

$$H_x \Delta l = I_\alpha \Delta l$$

where I_α is the linear current density. Then the magnetic field is related to the current by

$$H_x = I_\alpha = \frac{I}{a}$$

Now we have both V and I expressed in the field components, which leads to a relationship of transmission-line impedance and wave impedance as

$$Z = \frac{V}{I} = \frac{bE_y}{aH_x} = \eta_\varepsilon \frac{b}{a} \tag{32}$$

Eq.32 indicates that under the TEM mode approximation, the wave impedance of a microstrip only depends on the dielectric but is independent on the dimensions of a and b whereas the transmission-line impedance has an additional factor that is proportional to their ratio. When a approaches ∞, Z becomes zero.

While the Eq.32 provides a physical insight at the fundamental level how the wave impedance and transmission-line impedance are related for a specific configuration, we must point out that the formula in Eq.32 is too simple to be used for calculations of actual transmission-line impedances. For practical applications, various formulas for microstrip line are proposed to take the edge effects into account [11].

For comparison, the lowest EM wave mode for a coaxial cable is a perfect TEM mode (no edge effects in the transverse plane since the fields are completely enclosed inside the cable) and its characteristic impedance is given by [9]

$$Z = \frac{\eta_\varepsilon}{2\pi} \ln\left(\frac{r_o}{r_c}\right)$$

here r_c and r_o are the radii of the center and outer conductors respectively. We point out that the term "perfect" used here for a TEM mode refer to a condition where no fringing effect occurs. To have a mathematically perfect TEM mode, in addition to the absence of fringing effects, the conductors need to be PEC as well. The reason is the following. The general boundary condition requires the tangential components of electric field be continuous across a boundary [3-5]. In our case, it means the z-directed electric field on the conductor's surface, denoted as E_z^c, must be the same as E_z which is on the dielectric side and is the quantity that must be zero by definition for TEM mode. According to Ohm's law, $J_z = \sigma E_z^c$, only when $\sigma \to \infty$ E_z^c (hence E_z) vanishes since J_z is finite to support the magnetic field.

In summary, by the definition of transmission line used in this article, the dominant mode of any transmission line is either a perfect TEM mode or a quasi-TEM mode. For this reason, the TEM mode is also referred to as transmission-line mode in the literature [7, Sec.2.6].

4.3 Rectangular Waveguide

Now if we add two more parallel conductor walls to the structure in Fig. 5, we have a rectangular waveguide illustrated in Fig. 6. Unlike in the case of microstrip line where the edge effects can be dealt with as a correction factor, the physical change in waveguide structure fundamentally changes the wave behaviors. This is because a constant E_y in the TEM mode does not satisfy the boundary condition at two conductor walls at $x = 0$ and a planes. Consequently, one of the field components E_z or H_z must be nonzero, corresponding to two additional modes, TE mode ($E_z = 0$) and TM mode ($H_z = 0$) respectively. We analyze them separately below.

TM mode

In this mode, E_z is nonzero. Again, we still only need to consider its x, y dependance. Using Eq.6, we write $E_z(x, y)$ as

$$E_z(x, y) = (C_1 \sin k_x x + C_2 \cos k_x x)\big(C_3 \sin k_y y + C_4 \cos k_y y\big) \tag{33}$$

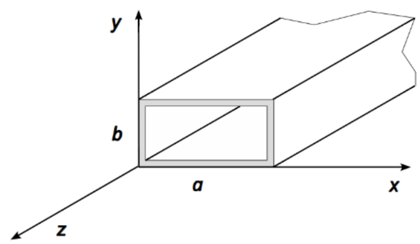


Figure 6 Rectangular waveguide

The boundary condition at $x = 0$ for E_z requires

$$E_z(x=0) = C_2\big(C_3 \sin k_y\, y + C_4 \cos k_y\, y\big) = 0 \tag{34}$$

Eq.34 holds for any y, hence we must have $C_2 = 0$. Similarly, that $E_z(y=0) = 0$ is true for any x leads to $C_4 = 0$. Thus Eq.33 reduces to

$$E_z(x, y) = C_1 C_3 \sin k_x\, x \sin k_y\, y \tag{35}$$

Further, $E_z(x=a) = 0$ and $E_z(y=b) = 0$ require

$$k_x = \frac{m\pi}{a} \quad k_y = \frac{n\pi}{b} \tag{36}$$

where m and n are non-zero integers. In this mode both m and n must be nonzero, otherwise E_z is 0 according to Eq.35. This fact makes TM mode different from TE mode as shown below.

TE mode.

In this case H_z is nonzero. Write a general solution for $H_z(x, y)$ in the same form as Eq.33,

$$H_z(x, y) = (C_1' \sin k_x\, x + C_2' \cos k_x\, x)\big(C_3' \sin k_y\, y + C_4' \cos k_y\, y\big) \tag{37}$$

Application of boundary conditions is less straightforward in this case because $H_z(x, y)$ does not vanishes at the conductor walls. We first need to relate H_z to E_x and E_y. From Eq.10a and Eq.10b we have

$$E_x = \frac{j\omega\mu}{k_z^2 - k_\omega^2}\frac{\partial H_z}{\partial y} = \frac{j\omega\mu}{k_z^2 - k_\omega^2}(C_1' \sin k_x x + C_2' \cos k_x x)\big(C_3' \, k_y \cos k_y \, y - C_4' \, k_y \sin k_y \, y\big) \qquad (38)$$

$$E_y = \frac{-j\omega\mu}{k_z^2 - k_\omega^2}\frac{\partial H_z}{\partial x} = \frac{-j\omega\mu}{k_z^2 - k_\omega^2}(C_1' \, k_x \cos k_x \, x - C_2' \, k_x \sin k_x \, x)\big(C_3' \sin k_y \, y + C_4' \cos k_y \, y\big) \qquad (39)$$

Using the same argument as for TM mode, the boundary condition on E_x at $y = 0$ plane leads to $C_3' = 0$ and on E_y at $x = 0$ plane leads to $C_1' = 0$. Thus H_z in Eq.37 reduces to

$$H_z(x, y) = C_2' C_4' \cos(k_x \, x) \cos(k_y \, y) \qquad (40)$$

And E_x and E_y become:

$$E_x = \frac{-j\omega\mu k_y}{k_z^2 - k_\omega^2} C_2' C_4' \cos(k_x \, x) \sin(k_y \, y) \qquad (41a)$$

$$E_y = \frac{j\omega\mu k_x}{k_z^2 - k_\omega^2} C_2' C_4' \sin(k_x \, x) \cos\,(\,k_y \, y) \qquad (41b)$$

Applying $E_x(y = b) = 0$ and $E_y(x = a) = 0$ in Eqs. 41a and b lead to the same requirements on k_x and k_y as in Eq.36

$$k_x = \frac{m\pi}{a} \quad k_y = \frac{n\pi}{b}$$

again m and n are nonzero integers. But now one of them can be 0 because of the cosine functions in Eq.40 as opposed to sine functions in Eq.35.

Now we have shown that the allowed propagation mode, introduced in Sec. 3.2, for a rectangular waveguide is either TE_{mn} or TM_{mn} mode. The corresponding wave pattern is a standing wave with the number of nodes (the points where the wave function is 0) being $m - 1$ and $n - 1$ in the x and y directions respectively. Obviously, there is an infinite number of such modes. In contrast, for TEM mode as discussed in Sec.2 both k_x and k_y must be 0, suggesting a uniform wave pattern. As a result, in an ideal condition assumed in this note there is no higher-order TEM mode. In the literature there are discussions on higher order TEM modes when the media or structure are nonuniform.

The TE mode is especially important in practical waveguide specifications as explained in the next paragraph. We complete this topic by providing the expressions for H_x and H_y for this mode. The $\boldsymbol{E}$ fields and H_z are already given in Eqs. 41 and 40.

$$H_x = \frac{-jk_z k_x}{k_z^2 - k_\omega^2} C_2' C_4' \sin(k_x x)\cos(k_y y) \tag{42a}$$

$$H_y = \frac{-jk_z k_y}{k_z^2 - k_\omega^2} C_2' C_4' \cos(k_x x)\sin(k_y y) \tag{42b}$$

The derivations are straightforward using Eqs. 10c and 10d and Eq.40.

Cutoff frequencies of TE and TM modes

Substituting Eq. 36 in Eq.15 gives the cutoff frequency for TE_{mn} and TM_{mn} modes:

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{43}$$

For standard rectangular waveguide WR series, the width a is chosen to be twice the height b, i.e. $a = 2b$. Then the modes for the first three lowest cutoff frequencies are TE_{10}, TE_{01} (also TE_{20}) and TE_{11} (also TM_{11}). Thus, to guarantee a single mode operation the usable frequency range is specified as f_c(TE_{10}) to $2f_c$(TE_{10}) with some margins on both sides.

Wave Impedance of TE and TM Modes

Next, we derive the wave impedance for the TE and TM modes. To that end, we need to find the expressions for a pair of transverse components of either E_x and H_y or E_y and H_x. Using Eq.10 for TM mode ($H_z = 0$) we have E_x and H_y as:

$$E_x = \frac{jk_z}{k_z^2 - k_\omega^2}\frac{\partial E_z}{\partial x}$$

$$H_y = \frac{j\omega\varepsilon}{k_z^2 - k_\omega^2}\frac{\partial E_z}{\partial x}$$

Thus

$$\frac{E_x}{H_y} = \frac{E_y}{H_x} = \frac{k_z}{\omega\varepsilon}$$

The expressions for E_y and H_x are left for the reader to verify. Using the expression for k_z in Eq.16 and Eq.26, we have

$$\eta_{\mathrm{TM}} = \frac{k_z}{\omega\varepsilon} = \eta_\varepsilon\sqrt{1 - \left({}^{\omega_c}/_{\omega}\right)^2} \tag{44}$$

By the same algebra we get the wave impedance for TE mode

$$\eta_{\mathrm{TE}} = \frac{\omega\mu}{k_z} = \eta_\varepsilon \frac{1}{\sqrt{1 - \left({}^{\omega_c}/_{\omega}\right)^2}} \tag{45}$$

Unlike the TEM mode where the wave impedance is independent of frequency the wave impedances for TE and TM modes are dependent on the frequency through the same mechanism discussed in the context of dispersion in Section 3. When $\omega \to \omega_c,\ \eta_{\mathrm{TE}} \to \infty$ and $\eta_{\mathrm{TM}} \to 0$ respectively. We make further remarks regarding the frequency dependence of wave impedance as follows.

1. It is fundamentally different from the frequency dependence of the correction terms in the formulas for transmission line impedances.
2. As shown in Fig.2 this dependence becomes less sensitive as frequency moves away from the cutoff frequency. Commercial waveguides are generally specified for operations at the dominant mode with a frequency range roughly from 20% to 80% above the cutoff frequency (1.2f_c to 1.8f_c).
3. When connecting two transmission lines the tolerance of the transmission-line impedances should be considered if a critical VSWR spec (a measure of discontinuity at the connection point) is required. In principle the same is true for waveguides in terms of wave impedance since it is a measure of the relative strength of an EM field. In practice, however, the wave impedance of a waveguide is generally not specified for the following reason: According to Eq.44, two factors (other than frequency) that affect the wave impedance are the medium (η_ε) and dimensions (via ω_c and Eq.43). For the former, most practical waveguides operate in the air, hence $\eta_\varepsilon = \eta_0$. And for the latter, the machining capability can easily exceed the required tolerance for waveguides with an operation frequency up to tens of Gigahertz which corresponds to a waveguide dimension in the range of centimeter. As a result, manufacturers generally do not make efforts to provide specifications for wave impedance.
4. Some advanced VNAs (vector network analyzer) offer measurement capability for waveguide components. In the corresponding calibration process characteristic impedance is a required parameter. In those cases, a default value (usually 1) should be used based on the manufacture's recommendation. Interested readers can find more detailed descriptions on this calibration technique in a separate note "TRL VNA Calibration Technique Using Waveguide Standards".

5. While the term of wave impedance and characteristic impedance are sometimes used interchangeably in the literature, in stricter terminology used in this article, the wave impedance, defined as E/H, is applicable to both waveguides and transmission lines whereas the characteristic impedance which is defined as V/I is not applicable to waveguides. To understand this, in addition to the TEM-mode argument outlined earlier, we can see the problem from another angle: in our narrow definition, a waveguide has one conducting wall. Thus the voltage, which is the potential difference between two spatial points, does not exist in the waveguide.

Since almost all commercial waveguides are specified for the operation at TE_{10} mode, the following is a table of the results obtained in this section for this mode. a in the Field expression is the waveguide width.

Fields/Parameters	**Expressions**
Fields	$E_y = E_0 \sin(\pi x/a)$ $H_x = (-E_0/\eta_{\mathrm{TE}}) \sin(\pi x/a)$ $H_z = (j\pi E_0/\omega\mu a) \cos(\pi x/a)$ $E_x = E_z = H_y = 0$
Wave Impedance	$\eta_{\mathrm{TE}} = \frac{\omega\mu}{k_z} = \eta_\varepsilon \frac{1}{\sqrt{1-\left({}^{\omega_c}/_{\omega}\right)^2}}$
Cutoff Frequency	$\omega_c = \frac{\pi}{a\sqrt{\mu\varepsilon}}$
Velocity in the Unbounded Medium	$v = \frac{1}{\sqrt{\mu\varepsilon}}$
Wavelength in the Unbounded Medium	$\lambda = \frac{v}{f}$
Phase Velocity	$v_p = \frac{\omega}{k_z} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$

Group Velocity	$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_z} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$
Waveguide Wavelength	$\lambda_g = \frac{2\pi}{k_z} = \frac{\lambda}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$

Table 1 Fields and parameters for rectangular waveguide TE_{10} mode.

5. Summary

We conclude this article with a summary.

1. **Waveguides and transmission Lines**. We made a distinction between them: a waveguide is a single conductor tube whereas a transmission line has at least two conductors. This structural difference is the origin for their different behaviors in microwave circuit applications.
2. **Wave number**. The characteristics of a guided wave is characterized by its wave number k_z in Eq. 14:

$$k_z^2 = \omega^2 \mu \varepsilon - \left(k_x^2 + k_y^2\right)$$

3. **TEM mode**. In this mode, $k_x = 0$ and $k_y = 0$, which has three important consequences: i. Constant $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in the transverse plane; ii. Zero cutoff frequency; iii. A simple relationship between wavelength and frequency, $\lambda f = v$, where v is the wave speed. Transmission lines operate in the TEM or quasi-TEM mode.
4. **Boundary condition and TE/TM mode**. The boundary condition $E_t = 0$ (Eq.13) at an interface between a perfect conductor and a dielectric medium is in conflict with the field pattern in the TEM mode described in item 3. As a result, the TEM mode is not allowed in a waveguide. This is the reason that a waveguide has a non-zero cutoff frequency.
5. **Wave Impedance and Characteristic Impedance of Transmission Line.** In our narrow and stricter definitions, the wave impedance is well defined for both waveguides and transmission-lines and is a useful parameter in analysis but has limited usage in practical circuit applications. In comparison, the characteristic impedance, which is critically important in practice, is only applicable to transmission lines.

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Appendix

The Method of Variable Separation for Three-Dimensional Wave Equations

Consider the three-dimensional wave equation [1,2]

$$\nabla^2 U(\vec{r},t) = \frac{1}{v^2}\frac{\partial^2 U(\vec{r},t)}{\partial t^2} \tag{1}$$

where $\vec{r}$ represents position vector and t is time. v is the wave velocity and a constant. In physics and engineering we are mostly interested in a solution that has a sinusoidal function for the time dependence. In phasor notation [2,3], this means $U(\vec{r},t)$ can be expressed as $U(\vec{r},t) = u(\vec{r})e^{j\omega t}$, with ω being angular frequency. Then Eq. 1 becomes

$$\nabla^2 u(\vec{r}) = -\beta^2\, u(\vec{r}) \tag{2}$$

where $\beta = \omega/v$.

In Cartesian coordinates, Eq. 2 can be expanded as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\beta^2\, u(x,y,z) \tag{3}$$

This is Helmholtz equation. We are seeking a solution that is in the form of separation of variables, that is,

$$u(x,y,z) = X(x)Y(y)Z(z) \tag{4}$$

In the following we simply attempt to show that a solution in the form of Eq.4 is allowed by Eq.3 rather than making a general statement about solutions to Eq.3. This form of solution is ultimately proven appropriate by the evidence that conclusions derived from it predict physics correctly.

Substituting Eq.4 into Eq.3 and then dividing the new equation by XYZ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\beta^2 \tag{5}$$

For Eq.5 to generally hold true, each term on the left-hand side much be a constant. Denoting them as $-k_x^2, -k_y^2, -k_z^2$ respectively, we have three equations for X, Y and Z as

$$X'' = -\,k_x^2 X, \qquad Y'' = -k_y^2 Y, \qquad Z'' = -k_z^2 Z \tag{6}$$

Eq.6 is a second order ordinary differential equation. Its general solution in complex number is[1]

$$X(x) = Ae^{jk_x x} + Be^{-jk_x x} \tag{7}$$

where A and B are a constant. When the time-dependent factor $e^{j\omega t}$ is included the two terms in Eq.7 represent waves traveling in opposite directions [1,3]. Using Euler formula, $X(x)$ in Eq.7 can also be expressed in trigonometric functions

$$X(x) = A'\cos(k_x x) + B'\sin(k_x x) \tag{8}$$

Thus, a general solution to Eq.3 can be expressed

$$u(x,y,z) = \left(Ae^{jk_x x} + Be^{-jk_x x}\right)\left(Ce^{jk_y y} + De^{-jk_y y}\right)\left(Ee^{jk_z z} + Fe^{-jk_z z}\right) \tag{9}$$

or

$$u(x,y,z) = (\ A'\cos k_x x + B'\sin k_x x)\ (\ C'\cos k_y y + D'\sin k_y y)\ (\ E'\cos k_z z + F'\sin k_x z) \tag{10}$$

Constants $A(A')$ through $F(F')$ for a specific engineering problem are determined by a set of boundary conditions. Furthermore, according to Eq. 5, k_x, k_y and k_z must satisfy

$$k_x^2 + k_y^2 + k_z^2 = \beta^2 \tag{11}$$

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